Mathieu Cubes

Mapping Mathieu groups M12 and M24 to 3x3x3 and 4x4x4 Cubes

Click a link below to play with M12 or M24 puzzle (may take a while to download)

Mathieu Cube M12 – 3x3x3 Cube

http://www.randelshofer.ch/rubik/virtualcubes/rubik/3x_scripts/m12/index_enVE.html

M12 Cube

Original design 2010
by André Boulouard
and Walter Randelshofer

M12 Cube Texture

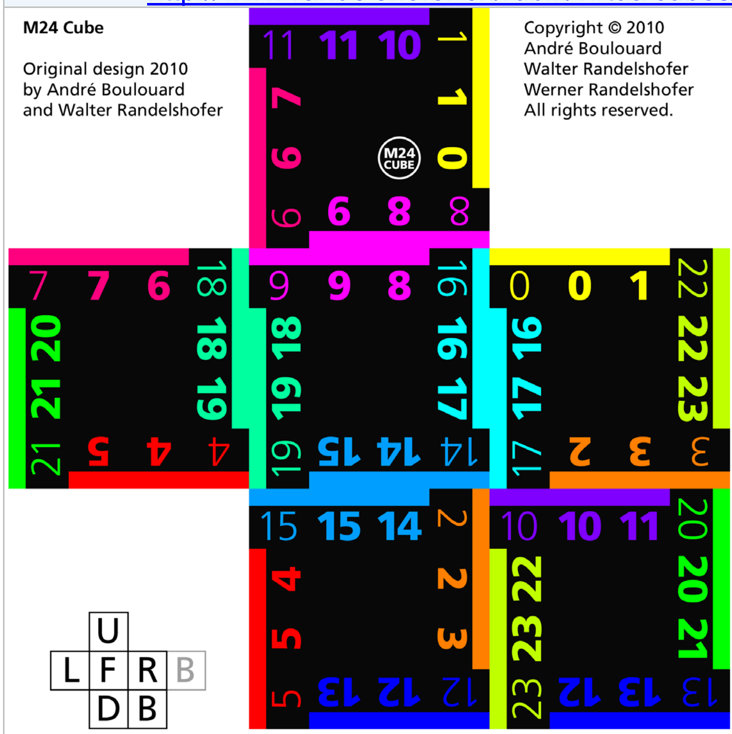
Virtual M12 Cube

Mathieu Cube M24 – 4x4x4 Cube

http://www.randelshofer.ch/rubik/virtualcubes/revenge/4x_scripts/4x_m24/index_enVE.html

M24 Cube

Original design 2010
by André Boulouard
and Walter Randelshofer



M24 Cube Texture

Virtual M24 Cube

Notes

1- All textures shown in the present document are copyright protected under the Creative Commons License terms.

2- A computer program has been written in JavaScript to find optimal sequences of M12 and M24 groups and generate databases of such sequences. This program is based on feature article 'Rubik's Cube Inspired Puzzles Demonstrate Math's "Simple Groups"', by Igor Kriz and Paul Siegel, which was printed in the July 2008 issue of Scientific American.

3- Databases of optimal sequences have been used to design M12 and M14 puzzles and animate virtual Mathieu cubes. The puzzles can be played with at: http://www.randelshofer.ch/.

Useful Links

Mathieu Groups – Useful Links	
http://en.wikipedia.org/wiki/Mathieu_group	http://www.math.uic.edu/~ronan/mathieu
http://selliott.org/puzzles/m12	http://forum.projecteuler.net/viewtopic.php?f=26&t=2171
http://www.gap-system.org/	http://brauer.maths.qmul.ac.uk/Atlas/v3/spor/
http://cube.stanford.edu/class/files/rokicki_cubecomp.pdf	
http://www.scientificamerican.com/article.cfm?id=simple-groups-at-play	
http://www.scientificamerican.com/article.cfm?id=what-is-a-sporadic-simple	
http://www.scientificamerican.com/article.cfm?id=puzzles-simple-groups-at-play	
http://www.scientificamerican.com/article.cfm?id=the-cycle-notation-and-m12	
http://www.scientificamerican.com/media/inline/2008-07/puzzles/m12.html (Click 'HELP' for hints)	
http://www.scientificamerican.com/media/inline/2008-07/puzzles/m24.html (Click 'HELP' for hints)	
http://motls.blogspot.com/2010/04/k3-and-mathieu-m24-group-new-moonshine.html	
http://oskarvandeventer.nl/M12/Developing_Topsy_Turvy_and_Number_Planet.ppt.pdf	
http://oskarvandeventer.nl/M12/Developing_Topsy_Turvy_and_Number_Planet.doc.pdf	
http://oskarvandeventer.nl/M12/	
http://cubezzz.dyndns.org/drupal/?q=node/view/88	
http://homepages.wmich.edu/~drichter/mathieu.htm	

For those interested in computer cubing, see Tomas Rokicki's informative paper on "Computers and the Cube".

Quotations

"There are almost as many different constructions of M_{24} as there have been mathematicians interested in that most remarkable of all finite groups."
-- John H. Conway

"Group theory, and specific groups, are full of non-intuitive things; I think this showed up repeatedly during the classification of the sporadic groups."
-- Tomas Rokicki

Introduction

Mathieu groups were the first sporadic simple groups discovered. They are usually denoted by the symbols M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, and can be thought of respectively as permutation groups on sets of 11, 12, 22, 23 or 24 objects (or points). The Mathieu groups are fascinating to many group theorists as mathematical anomalies.

As there are 12 edges on a 3x3x3 Rubik's cube, a map from the M12 group to permutations of cube edges can be achieved. The same holds for mapping the M24 group to permutations of the 24 edges of a 4x4x4 Revenge cube.

A numbering system has been designed to ease positioning edges of Mathieu Cubes, which show the same numbers as neighbouring corner(s). Edges are the only pieces that move, whereas corners and centers stay in their initial positions.

Databases

Databases of 95,039 (up to 29 moves) and 86,882 (up to 20 moves) optimal sequences have been generated for the M12 and M24 puzzles, respectively. Sequences are encoded by an 'home-brewed' Base90 JavaScript encoder to lower file size and thus reduce download time, which is a key feature of on-line apps. By clicking the appropriate link to the puzzle, sequences are downloaded, decoded and executed *only once* at program start, to give arrays of 'ready-to-go' sequences and puzzle states. Decoded sequences and states are further used throughout the play to compute permutations dynamically and display 'cheat sequences' to guide the puzzle player.

Base-90 Encoder/Decoder

The ASCII code includes 95 printable characters. From these, 5 characters have been left out because of possible JavaScript malfunction: space ( ), double quote ("), ampersand (&), less-than sign (<) and antislash (\). All remaining 90 characters can then be used to encode strings of characters into *variable-length* words. For example, sequence "SRRSLLSRRSLLLSRRSLLSRR", which is a 24-character string, is coded as an 8-character word ",kF?Az", with a net coding gain of 3.

M12 Group

The order (number of elements) of M12 is 95,040. The entire group can be generated by 2 permutations. We use the same generators as those of the SciAm feature article, namely I and M, defined in standard cycle notation as follows:

I = (1,12)(2,11)(3,10)(4,9)(5,8)(6,7) -- 6 2-cycles or 6(2c)

M = (1)(2,3,5,9,8,10,6,11,4,7,12) -- 11-cycle or (11c)

where permuted elements are numbered from 1 to 12.

Using a Breadth First Search (BFS), *all* optimal sequences of the M12 group have been computed up to depth 29, which is the group God's number, ie. the distance of the farthest position from start, or equivalently the diameter of the Cayley graph of the group.

Permutations can be sorted out by type of cycle structure. There are 13 distinct cycle structures in the M12 group. The maximum order of any element in M12 is 11 (orders 7 and 9 are lacking), as shown in the table below:

M12 Group Conjugacy Classes – Set of Generators <I, M>					
Order	**No. elements**	**Conjugacy**	**Cycle Structure**	**Index**	**Moves***
1 = 1	1 = 1	1 trivial class	–	–	–
2 = 2	$891 = 3^4 \cdot 11$	2 classes (not power equivalent)	4(2c)	1	27m*
			6(2c)	2	27m*
3 = 3	$4{,}400 = 2^4 \cdot 5^2 \cdot 11$	2 classes (not power equivalent)	3(3c)	3	28m*
			4(3c)	4	27m*
$4 = 2^2$	$5{,}940 = 2^2 \cdot 3^3 \cdot 5 \cdot 11$	2 classes (not power equivalent)	2(4c)	5	28m*
			2(2c) 2(4c)	6	28m*
5 = 5	$9{,}504 = 2^5 \cdot 3^3 \cdot 11$	1 class	2(5c)	7	29m*
$6 = 2 \cdot 3$	$23{,}760 = 2^4 \cdot 3^3 \cdot 5 \cdot 11$	2 classes (not power equivalent)	(2c) (3c) (6c)	8	28m*
			2(6c)	9	28m*
$8 = 2^3$	$23{,}760 = 2^4 \cdot 3^3 \cdot 5 \cdot 11$	2 classes (not power equivalent)	(2c) (8c)	10	29m*
			(4c) (8c)	11	29m*
$10 = 2 \cdot 5$	$9{,}504 = 2^5 \cdot 3^3 \cdot 11$	1 class	(2c) (10c)	12	28m*
11 = 11	$17{,}280 = 2^7 \cdot 3^3 \cdot 5$	2 classes (power equivalent)	(11c)	13	29m*

Notes on M12

1- As 95,040 = 12!/7!, M12 and symmetry group S7 can be used to build symmetry group S12: S12 = S7 x M12, where the representation is unique for any element of S12.

2- The maximum number of moves to set the puzzle to a particular position in a given conjugacy class is 0, 1 or 2 moves less than the group diameter, that is 29, 28 or 27 moves. This shows that the 'diameter' of any conjugacy class is very close to the entire group diameter. We don't know if this does also hold for M24, though.

3- There are twelve antipodes in M12, ie. positions that are at largest distance from solved. The twelve 29-move sequences are given below:
"I M I M I M4 I M4 I M I M I M2 I M2 I M4",
"I M I M4 I M2 I M I M I M4 I M3 I M I M3",
"I M I M4 I M3 I M I M I M2 I M4 I M I M3",
"I M2 I M2 I M2 I M2 I M I M I M4 I M3 I M3",
"I M2 I M2 I M2 I M2 I M I M I M4 I M6 I",
"I M2 I M2 I M5 I M I M I M5 I M6",
"I M3 I M2 I M4 I M I M2 I M I M4 I M4",
"I M5 I M3 I M I M I M4 I M2 I M I M2 I M",
"M I M I M I M3 I M4 I M I M I M2 I M2 I M4",
"M I M I M I M3 I M4 I M I M I M2 I M5 I M",
"M I M2 I M2 I M2 I M I M I M4 I M3 I M I M3",
"M I M2 I M2 I M2 I M I M I M4 I M4 I M2 I M"

M24 Group

The order (number of elements) of M24 is 244,823,040. The entire group can can be generated by 3 permutations. We use the same generators as those of the SciAm feature article, namely L, R and S, defined in standard cycle notation as follows:

L = (0)(23,22,21,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2,1) -- 23-cycle or (23c)

R = (0)(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23) -- 23-cycle or (23c)

S = (0,1)(2,23)(3,4)(5,22)(6,11)(7,8)(9,10)(12,21)(13,14)(15,20)(16,17)(18,19) -- 12 2-cycles or 12(2c)

where permuted elements are numbered from 0 to 23.

It can be seen that L and R are inverse permutations and that S is its own inverse. So, when searching for optimal sequences, L should never be followed by R, R by L and S by S.

Using a Breadth First Search (BFS), a set of optimal sequences of the M24 group has been computed up to depth 23. Although the group God's number has not been computed yet, it has been estimated to be 40 – 45 by Jaap.

Permutations can be sorted out by type of cycle structure. There are 20 distinct cycle structures in the M24 group and the maximum order of any element in M24 is 23 (orders 9, 13, 16, 17, 18, 19, 20 and 22 are lacking), as shown in the table below:

M24 Group Conjugacy Classes – Set of Generators <L, R, S>				
Order	**No. elements**	**Conjugacy**	**Cycle Structure**	**Index**
1 = 1	1	1 trivial class	–	–
2 = 2	11,385 = $3^2 \cdot 5 \cdot 11 \cdot 23$	2^8, 1 class	8(2c)	1
	31,878 = $2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 23$	**2^{12}, 1 class**	**12(2c)**	2
3 = 3	226,688 = $2^7 \cdot 7 \cdot 11 \cdot 23$	3^6, 1 class	6(3c)	3
	485,760 = $2^7 \cdot 3 \cdot 5 \cdot 11 \cdot 23$	3^8, 1 class	8(3c)	4
4 = 2^2	637,560 = $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2^4 4^4$, 1 class	4(2c) 4(4c)	5
	1,912,680 = $2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2^2 4^4$, 1 class	2(2c) 4(4c)	6
	2,550,240 = $2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	4^6, 1 class	6(4c)	7
5 = 5	4,080,384 = $2^8 \cdot 3^3 \cdot 7 \cdot 11 \cdot 23$	5^4, 1 class	4(5c)	8
6 = 2 · 3	10,200,960 = $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2^2 3^2 6^2$, 1 class	2(2c) 2(3c) 2(6c)	9
	10,200,960 = $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2^4 4^4$, 1 class	4(6c)	10
7 = 7	11,658,240 = $2^{10} \cdot 3^2 \cdot 5 \cdot 11 \cdot 23$	7^3, 2 power equivalent classes	3(7c)	11
8 = 2^3	15,301,440 = $2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2 \cdot 4 \cdot 8^2$, 1 class	(2c) (4c) 2(8c)	12
10 = 2 · 5	12,241,152 = $2^8 \cdot 3^3 \cdot 7 \cdot 11 \cdot 23$	$2^2 10^2$, 1 class	2(2c) 2(10c)	13
11 = 11	22,256,640 = $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 23$	11^2, 1 class	2(11c)	14
12 = $2^2 \cdot 3$	20,401,920 = $2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2 \cdot 4 \cdot 6 \cdot 12$, 1 class	(2c) (4c) (6c) (12c)	15
	20,401,920 = $2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	12^2, 1 class	2(12c)	16
14 = 2 · 7	34,974,720 = $2^{10} \cdot 3^3 \cdot 5 \cdot 11 \cdot 23$	$2 \cdot 7 \cdot 14$, 2 power equivalent classes	(2c) (7c) (14c)	17
15 = 3 · 5	32,643,072 = $2^{11} \cdot 3^2 \cdot 7 \cdot 11 \cdot 23$	$3 \cdot 5 \cdot 15$, 2 power equivalent classes	(3c) (5c) (15c)	18
21 = 3 · 7	23,316,480 = $2^{11} \cdot 3^2 \cdot 5 \cdot 11 \cdot 23$	$3 \cdot 21$, 2 power equivalent classes	(3c) (21c)	19
23 = 23	21,288,960 = $2^{11} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	23, 2 power equivalent classes	(23c)	20

Notes on M24

1- It has been shown that the 15-puzzle can be solved in 43 'moves' or less. Although there is no clear connection between M24 and 15-puzzle groups, it happens that they nearly have the same diameter.

M24 and Fibonacci Numbers

In mathematics, the **Fibonacci numbers** are the numbers in the following integer sequence:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …

By definition, the first two Fibonacci numbers are 0 and 1, and each subsequent number is the sum of the previous two. Some sources omit the initial 0, instead beginning the sequence with two 1s. In mathematical terms, the sequence F_n of Fibonacci numbers is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}$$

with seed values

$$F_0 = 0 \text{ and } F_1 = 1$$

It happens that the number of optimal sequences of a given length of M24 12(2c) class (more or less) closely follows a Fibonacci sequence, at shown in the table below:

M24 Conjugacy Class 12(2c) – Fibonacci Numbers					
Sequence Length	Number of Sequences	Fib Numbers	Sequence Length	Number of Sequences	Fib Numbers
1	1	2	23	282	288
3	2	2	25	–	466
5	4	4	27	–	754
7	6	6	29	–	1,220
9	10	10	31	–	1,974
11	16	16	33	–	3,194
13	26	26	35	–	5,168
15	40	42	37	–	8,362
17	61	68	39	–	13,530
19	97	110	41	–	21,892
21	180	178	**43**	–	35,422

As the order of class 12(2c) is 31,878, it may be inferred from the table that the maximum sequence length of elements of the class could be about 43. From previous results on M12 and by applying heuristics, it may be further inferred that God's number of the class could be close to God's number of M24, so doesn't that lead to the conclusion that 43 would be a good candidate for God's number of M24?

Building Conjugacy Class 12(2c)

A conjugacy class of a group of permutations G can be build by conjugating a *single* representative of the class with elements of G, until the class is complete. If sequences of moves (words) are used instead of permutations, more than a single representative sequence should be used though, otherwise conjugated sequences will be too long. If a set of optimal sequences is already available, then they can be used as *seeds*. A seed is defined as an optimal sequence that can be *shifted*, *inverted* and *conjugated* to give additional short sequences that will ultimately complete the conjugacy class.

If we take for example seed sequence R S R R S L L L L S L L S R R R R S R R S, which is 21 moves long, we can shift it 21 times and conjugate all shifted versions to get new sequences and thus new puzzle states:

shift right 1 move : S R S R R S L L L L S L L S R R R R S R R
shift right 2 moves: R S R S R R S L L L L S L L S R R R R S R
etc…

By shifting and conjugating all seeds of class 12(2c), presently available up to length 23, it has been possible to show that there are no sequences longer than 45 moves in the class. As using seeds to build classes *usually* give sequences which are at most 2 moves longer than the diameter of the class, this would also give 43 as a possible candidate to God's number of M24. By applying more heuristics, as God's number of M12 is an odd number (29), it may well be the same for M24.